Möbius parametrizations of curves in $\mathbb{R}^n$

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Abstract. We use Ahlfors' definition of Schwarzian derivative for curves in euclidean spaces to present new results about *Möbius* or *projective parametrizations*. The class of such parametrizations is invariant under compositions with Möbius transformations, and the resulting curves are simple. The analysis is based on the oscillatory behavior of the associated linear equation $u'' + \frac{1}{4}k^2u = 0$, where $k = k(s)$ is the curvature as a function of arclength.

1. Introduction

In this paper we continue the work in [4] and [3] based on Ahlfors' definition of Schwarzian derivative for curves γ in euclidean space [1], where we developed sharp bounds on the real part of the Schwarzian which imply that γ is simple or unknotted. In [7] Kobayashi and Wada consider a Schwarzian and a conformal Schwarzian derivative for curves in arbitrary Riemannian manifolds. The two operators coincide when the metric is Einstein, and in the euclidean space one can recover from it Ahlfors' operator. In [6] the author defines the projective structure of curves in arbitrary manifolds based on what could be called *Möbius* or *projective parametrizations.* The purpose of this paper is to study Möbius parametrizations in $\mathbb{R}^n$ and offer new perspectives and results. The class of Möbius parametrizations will be preserved under composition with Möbius transformations in the range or in the parameter. In general, a curve of infinite length will not admit a Möbius parametrization globally, but rather only on subarcs described in terms of the arclength parameter s and the curvature $k(s)$ exactly as those on which some solution of the linear equation $u(s) + \frac{1}{4}k^2(s)u(s) = 0$ does not vanish. Also, any such portion will not intersect itself, a fact that by different means has been

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established in [6] in the general setting. As as consequence, we will show certain infinite ends of curves in $\mathbb{R}^n$ to be simple. The parametrization will have order of contact three with a canonically chosen parametrization of the osculating circle by a Möbius map. Finally, by piecing together the Möbius parametrizations of consecutive portions of a curve of infinite length we will establish a theorem that accounts for the number of zeros of the equation $u'' + pu = 0$ for an arbitrary continuous function p.

2. Möbius Parametrizations

Let $f : (a, b) \to \mathbb{R}^n$ be a C^3 curve with $f' \neq 0$, and let $X \cdot Y$ stand for the euclidean inner product of vectors X, Y in $\mathbb{R}^n$ and $|X|^2 = X \cdot X$. As was pointed out in [4], it is easy to see that the real part of Ahlfors' Schwarzian, defined by

$$S_1 f = \frac{f' \cdot f'''}{|f'|^2} - 3\frac{(f' \cdot f'')^2}{|f'|^4} + \frac{3}{2}\frac{|f''|^2}{|f'|^2}\,,$$

can be written in terms of the velocity $v = |f'|$ and the curvature k of the trace of f as

$$S_1 f = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2 k^2\,, \tag{2.1}$$

and that this expression is invariant under the Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$. Note that $S_1 f = Ss + \frac{1}{2}v^2 k^2$, where $s = s(x)$ is arclength as a function of $x \in (a, b)$. This definition coincides with the 0-part of the Schwarzian defined in [7] (see also [Lemma 2.1, 6]).

For a real valued function h with $h' \neq 0$, Sh is the usual Schwarzian

$$Sh = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2 .$$

We recall the addition formula

$$S(h \circ g) = (Sh \circ g)(g')^2 + Sg\,, \tag{2.2}$$

and in particular,

$$Sh = -Sg(h')^2 \quad , \quad g = h^{-1}\,, \tag{2.3}$$

because the identity has vanishing Schwarzian.

Our main result in [4] was:

Theorem A. *Let $p = p(x)$ be a continuous real-valued function on an open interval I such that any nontrivial solution of $u'' + pu = 0$ has at most one zero on I. Let $f : I \to \mathbb{R}^n$ be a C^3 curve with $f' \neq 0$. If $S_1 f \leq 2p$, then f is one-to-one on I and admits a spherically continuous extension to the closed interval, which is also one-to-one unless the trace of f is a circle, in which case $S_1 f \equiv 2p$.*

We give the following definition.

Definition 2.1. We say that $f : I \to \mathbb{R}^n$ is a Möbius parametrization, or M-parametrization, if $S_1 f = 0$. We say that f is an M-parametrization of γ if in addition $f(I) = \gamma$. The interval I may be a finite, semi-infinite or even equal to $\mathbb{R}$.

Theorem 2.2. *If $f : I \to \mathbb{R}^n$ is an M-parametrization then the image $f(I)$ is a simple curve. M-parametrizations are preserved under composition in range or domain with Möbius transformations.*

Proof. The first claim follows immediately from Theorem A since the equation $u'' = 0$ on any interval admits non-trivial solutions with at most one zero. For a different proof, see also [Theorem, p.2, 6]. For the second claim, let f be an M-parametrization. Because S_1 is invariant under Möbius transformations T of $\mathbb{R}^n \cup \{\infty\}$, then $T \circ f$ is also an M-parametrization. On the other hand, because of the chain rule

$$S_1(f \circ \sigma) = [(S_1 f) \circ \sigma](\sigma')^2 + S\sigma \,,$$

it follows that $f \circ \sigma$ remains an M-parametrization whenever $\sigma : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ is Möbius. □

Throughout the paper, $k = k(s)$ will denote a non-negative function defined for all $s \in \mathbb{R}$ and $\Gamma \subset \mathbb{R}^n$ a curve (of infinite length) with curvature $k(s)$. Let $\phi = \phi(s)$ be an arclength parametrization of Γ.

Theorem 2.3. *Let $\Gamma_1 = \phi((s_1, s_2))$ be an arc of Γ, $\infty \le s_1 < s_2 \le \infty$. Then there exists an M-parametrization of Γ_1 if and only if some solution of $u''(s) + \frac{1}{4}k^2(s)u(s) = 0$ does not vanish on (s_1, s_2).*

Proof. Suppose first there exists an M-parametrization f of an arc $\Gamma_1 \subset \Gamma$. Then

$$S_1 f = Ss(x) + \frac{1}{2}v^2 k^2(s(x)) = 0 \,. \tag{2.4}$$

Let $x = h(s)$ be the inverse of $s = s(x)$. Then, by (2.3),

$$Sh = -(Ss)(h')^2 = \frac{1}{2}k^2 \,. \tag{2.5}$$

It is standard that $u(s) = (h')^{-1/2} > 0$ is a solution of

$$u'' + \frac{1}{2}(Sh)u = 0 \,,$$

as desired.

On the other hand, let $u(s)$ be a solution of $u''(s) + \frac{1}{4}k^2(s)u(s) = 0$ which is positive on (s_1, s_2). Fix $s_0 \in (s_1, s_2)$ and let

$$x = h(s) = \int_{s_0}^{s} u^{-2}(t)dt \,. \tag{2.6}$$

A simple calculation shows that $Sh = \frac{1}{2}k^2(s)$, therefore

$$f(x) = \phi(s(x))$$

has
$$S_1 f = Ss + \frac{1}{2}v^2k^2 = -(Sh)(h')^{-2} + \frac{1}{2}v^2k^2 = 0$$
because $v = (h')^{-1}$. $\square$

Recall the curve Γ and a subarc $\Gamma_1 = \phi((s_1, s_2))$.

Theorem 2.4. *Suppose f is an M-parametrization of Γ_1 defined on some interval I. Then for some σ Möbius, $f \circ \sigma$ can be extended to $I_0 = \mathbb{R}$ as an M-parametrization of an arc Γ_2 with $\Gamma_1 \subset \Gamma_2 \subset \Gamma$. Furthermore, $\Gamma = \Gamma_2$ if and only if $k(s) = 0$ for all s and Γ is a line.*

Proof. Let $\lambda(x) = v^{-1/2}(x) = |f'(x)|^{-1/2}$. Then
$$\lambda'' + \frac{1}{2}(Ss)\lambda = 0\,,$$
which according to (2.4) corresponds to
$$\lambda''(x) = \frac{1}{4}\lambda^{-3}(x)k^2(s(x))\,. \tag{2.7}$$
Any positive solution of (2.7) is convex. There are two possibilities:

(i) λ has one critical point at some $x_0 \in (a, b)$;

(ii) λ is either increasing or decreasing on (a, b).

Suppose (i). By convexity, $\lambda(x)$ grows at least at linear rate when moving away from x_0 in either direction. This implies that both integrals
$$\int_a^{x_0} \lambda^{-2}(x)dx \quad , \quad \int_{x_0}^b \lambda^{-2}(x)dx \tag{2.8}$$
are finite. But $\lambda^{-2} = |f'|$, and therefore, Γ_1 is of finite length. Because $k(s)$ is continuous and defined for all s, we see from (2.7) that $\lambda'(x)$ has a limit as $x \to a, b$. This implies the same conclusion for $\lambda(x)$ itself. We conclude that the solution λ can be continued beyond both endpoints a, b. In light of (2.7) and the linear rate of growth of $\lambda(x)$, we see that $\lambda''(x)$ will remain integrable, showing that $\lambda'(x)$, and thus $\lambda(x)$, cannot become infinite at some finite x. This proves that λ can be continued for all x as a solution of (2.7). In other words, f can be extended to all of $\mathbb{R}$ as an M-parametrization of some larger arc $\Gamma_2 \subset \Gamma$ of finite length.

Suppose now (ii). The argument is the same in each case, and we assume that λ is decreasing. The previous analysis applied moving to the left shows that the solution λ can be continued to $-\infty$. In other words, f can be extended to $(-\infty, b)$ as an M-parametrization of some larger subarc of Γ. Because $\lambda > 0$ is decreasing, $\lim_{x\to b^-} \lambda(x) = c$ exists.

Suppose $c = 0$. Then, by convexity, $\lambda(x) \le m(b - x)$ for some $m > 0$ as $x \to b^-$. Hence
$$\int^b \lambda^{-2}(x)dx = \infty\,,$$

therefore Γ_1 extends infinitely long, in this case, $s_2 = \infty$. Consider $g = f \circ \sigma$ with

$$\sigma(y) = b - \frac{1}{y}\,.$$

Then g is an M-parametrization defined on $(0, \infty)$. Let μ be the solution of (2.7) corresponding to g. Then μ must remain decreasing, for otherwise

$$\int^b \mu^{-2}(y)dy < \infty\,,$$

contradicting the fact that $s_2 = \infty$. But then our previous analysis is applicable to the left end point $y = 0$, to conclude that the solution μ can be extended to $-\infty$. This proves that $g = f \circ \sigma$ can be defined on $\mathbb{R}$ as an M-parametrization of some arc $\Gamma_2 \subset \Gamma$.

Suppose now $c > 0$. Because λ is decreasing and convex, it follows that $\lim_{x\to b} \lambda'(x)$ exists. Therefore, (2.7) again can be continued to the right. It will either remain decreasing for all x, will tend to 0 at some $b_1 > b$ or will reach a point $x_0 > b$ where $\lambda(x_0) > 0$ and $\lambda'(x_0) = 0$. In the first case, f can be extended to an M-parametrization of an arc $\Gamma_2 \subset \Gamma$ that has infinite length in one direction. In the second case, we repeat the analysis above for the case when $c = 0$ to conclude that some Möbius reparametrization $f \circ \sigma$ is defined on all of $\mathbb{R}$ as an M-parametrization of a semi-infinite arc Γ_2. In the third case, f will be defined on all of $\mathbb{R}$ as an M-parametrization of some arc Γ_2 that has finite length.

We see from this discussion that the only way to have $\Gamma_2 = \Gamma$, that is, to have Γ_2 be extended infinitely long in both directions, is that the solution λ defined on all of $\mathbb{R}$ does not become increasing as $x \to \pm\infty$. Because λ is convex, this forces λ to be constant, that is, $k(s) = 0$ for all s. This finishes the proof. □

It is well known that the absolute cross-ratio of four points in $\mathbb{R}^n$, defined by

$$|(p_1, p_2, p_3, p_4)| = \frac{|p_1 - p_3|\,|p_2 - p_4|}{|p_2 - p_4|\,|p_2 - p_3|}\,,$$

is Möbius invariant (see [Ah, Proposition 2]). By appealing to the relationship between the operator S_1 and the infinitesimal distortion of cross-ratio ([Ah, p.14] or [ChG, Theorem A]), it follows that the restriction of any Möbius transformation of $\mathbb{R}^n$ to a line is an M-parametrization, that is, the restriction of Möbius transformations to lines provide M-parametrizations of circles.

Let f be an M-parametrization of an arc $\Gamma_1 \subset \Gamma$, and let C_p be the osculating circle to Γ_1 at a point $p = f(x_0)$. Let us identify $\mathbb{R}$ with the first axis in $\mathbb{R}^n$, and let T be any Möbius of $\mathbb{R}^n$ with the property that $T(\mathbb{R})$ maps into C_p with $T(0) = p$. Let σ be a Möbius map of the form

$$\sigma(x) = \frac{\alpha x}{1 + \beta x}\,.$$

It is easy to see that α, β can be chosen in a way that the M-parametrization $T_{f,\,p}$ of C_p given by $T_{f,\,p} = T \circ \sigma$ satisfies

$$T_{f,\,p}(0) = p \quad , \quad |T'_{f,\,p}(0)| = |f'(x_0)| \quad , \quad |T'_{f,\,p}|'(0) = |f'|'(x_0)\,. \tag{2.9}$$

We call $T_{f,\,p}$ the canonical M-parametrization of the osculating circle C_p.

Theorem 2.5. *The canonical M-parametrization $T_{f,\,p}$ of the osculating circle C_p satisfies*

$$|T'_{f,\,p}|''(0) = |f'|''(x_0)\,.$$

Proof. Because f and $T_{f,\,p}$ are M-parametrizations, the functions $v = |f'|$ and $w = |T'_{f,\,p}|$ have

$$\left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 = -\frac{1}{2}v^2 k_{\Gamma_1}^2\,,$$

$$\left(\frac{w'}{w}\right)' - \frac{1}{2}\left(\frac{w'}{w}\right)^2 = -\frac{1}{2}w^2 k_{C_p}^2\,.$$

But $v(x_0) = w(0)$, $\;v'(0) = w'(0)$ by (2.9), and because $k_{\Gamma_1} = k_{C_p}$ at p, it follows that $v''(x_0) = w''(0)$, as claimed. □

3. Infinite Ends

In this section we will consider curves of infinite length that allow a Möbius parametrization. An elementary way to tell than an infinite end γ of a curve is simple is that

$$\int_\gamma k(s)ds < \pi\,.$$

The integral represents the length of the curve L in $\mathbb{S}^n$ traced out by the unit tangent vector. The estimate above forces L to stay within one hemisphere, that is, $(d/ds)[\gamma \cdot \hat{d}\,] > 0$ for some unit vector $\hat{d}$. This implies that γ is simple.

In light of Theorem 2.4, with the exception of lines, curves admitting an M-parametrization can extend infinitely long only in one direction. Simple models are based on curvature functions of the form

$$k(s) \;=\; \frac{c}{s^\alpha}\,,$$

defined for $s > 0$. Here $c, \alpha > 0$ are constants.

Theorem 3.1. *Let $\gamma(s)$ be an arc-length parametrization of a curve in $\mathbb{R}^n$ defined for $0 < s < \infty$ with curvature $k(s)$. If*

$$k(s) \;\le\; \frac{1}{s}$$

then $\gamma(s)$ admits an M-parametrization and is therefore simple.

Proof. The solutions of $u'' + \frac{1}{4s^2}u = 0$ are of the form $u(s) = \sqrt{s}(a + b\log(s))$. In particular, $u_0(s) = \sqrt{s}$ is non-vanishing. Since $k(s) \le 1/s$, there exists a positive solution of $u'' + \frac{1}{4}k^2 u = 0$, which shows that γ admits an M-parametrization, and is hence simple.

□

Note that in the case when $k(s) = 1/s$ the integral

$$\int_0^\infty k(s)ds$$

is divergent at both 0 and ∞. It is interesting that the equation $u'' + \frac{c}{4s^2}u = 0$ becomes oscillatory whenever $c > 1$ (see, *e.g.*, [5, Theorem 7.1). Planar curves with such a curvature function will remain simple (see below) but will not admit a global M-parametrization. It follows again by Sturm comparison that, for $\alpha > 1$, some infinite end of a curve with $k(s) = c/s^\alpha$ will admit an M-parametrization, whereas for $\alpha < 1$ this will never be the case.

The planar curves with $k(s) = c/s$ can be found explicitly, and are given after rotation and translation, by

$$x(s) = \frac{s}{\sqrt{1+c^2}}\left[\cos(c\log(s)) + c\sin(c\log(s))\right],$$
$$y(s) = \frac{s}{\sqrt{1+c^2}}\left[\sin(c\log(s)) - c\cos(c\log(s))\right].$$

They are simple because $x(s)^2 + y^2(s) = s^2/(1+c^2)$.

Perhaps one would be tempted to believe that any curve with $k(s) \sim c/s$, $s > 0$, is simple. But there is a curve with $c_1/s \le k(s) \le c_2/s$ which is not simple. In the example above, it is probably the very regular behavior of the curvature that prevents self-intersections. The example is given by

$$\gamma(t) = e^{at}(1 + e^{it}),\ a > 0.$$

Then $\gamma(t) = 0$ for $t = (2k+1)\pi$. The parametrization is regular in the sense that $\gamma'(t) = e^{at}(a + (a+i)e^{it}) \ne 0$ because $|a + (a+i)e^{it}| \ge |a+i| - a > 0$. We see that $e^{-at}|\gamma'(t)|$ is 2π-periodic, hence $s(t) \sim ce^{at}$ as $t \to \infty$. Similarly, one can see that $e^{-at}k(t)$ is 2π-periodic. In addition, a simple calculation shows that $\gamma''(t)/\gamma'(t)$ is real iff

$$ia(a+i)e^{it} + i(a^2+1)$$

is real. But $|\mathrm{Im}\{\, ia(a+i)e^{it} + i(a^2+1)\}| > a^2 + 1 - a\sqrt{a^2+1} > 0$, showing that $k(t)$ never vanishes. We conclude that for some positive constants c_1, c_2

$$\frac{c_1}{s} \le k(s) \le \frac{c_2}{s}.$$

One can produce many examples of infinitely long ends of M-parametrizable curves by simple choosing *any* decreasing convex function $\lambda(x)$ defined for all x. Equation (2.7) together with the relation $s(x) = \int_{-\infty}^x \lambda^{-2}(y)dy$ provide the arclength and corresponding curvature function.

4. Oscillation

Let u_0 be a non-trivial solution of $u'' + \frac{1}{4}k^2 u = 0$, and let $\{s_n\}$, be the sequence of its zeros. This sequence may be unbounded from below or from above. Because u_0 has constant sign on any interval $J_n = (s_n, s_{n+1})$, it follows that $\Gamma_n = \phi(J_n)$ is M-parametrizable. In fact, the M-parametrization provided by Theorem 2.3 via equation (2.6) will have x range from $-\infty$ to ∞. We can think of M-parametrizations defined on consecutive copies of $\mathbb{R}$ that cover Γ. Our motivation here is to piece them together by precomposing with the exponential map $x = e^{it}$. This function takes $\mathbb{R}$ onto the circle infinitely often, and by means of a Möbius transformation, each copy of the circle can be identified with a different copy of $\mathbb{R}$. The resulting construction will provide a single parametrization $\psi : \mathbb{R} \to \Gamma$ of the entire curve Γ, in which the only contribution to $S_1\psi$ will come from the exponential. Thus $S_1\psi = S_1(e^{it}) = 1/2$. Equations (2.4) and (2.5) will now read

$$Ss(t) + \frac{1}{2}v^2k^2 = \frac{1}{2} \quad , \quad Sh = \frac{1}{2}k^2 - \frac{1}{2}(h')^2 \,,$$

and the function $w = (h')^{-1/2}$ will be a (positive) solution of $w'' + \frac{1}{4}k^2 w = \frac{1}{4}w^{-3}$.

We have found it easiest for the presentation to start from this result.

Lemma 4.1. *Let $\lambda = \lambda(x) \neq 0$ be a solution of*

$$\lambda'' = \frac{1}{4}\lambda^{-3} \tag{4.1}$$

on some interval $I \subset \mathbb{R}$. Then λ extends to a non-vanishing solution of (4.1) on all of $\mathbb{R}$ and

$$\int_{-\infty}^{\infty} \lambda(x)^{-2} dx = 2\pi \,.$$

Proof. It is no loss of generality to assume that $0 \in I$ and that $\lambda(0) > 0$. Upon multiplication by λ', equation (4.1) can be integrated directly, and one finds that

$$\lambda(x) = \frac{1}{2a}\sqrt{4a^4 + (x-b)^2}\,,$$

where the constants a, b are chosen to match up with the initial conditions of λ at $x = 0$. A direct integration gives now the conclusion of the lemma. □

Theorem 4.2. *Let $p = p(s)$ be defined and continuous for $s \in \mathbb{R}$. Let $w \neq 0$ be a solution of*

$$w'' + pw = \frac{1}{4}w^{-3} \tag{4.2}$$

on some interval $J \subset \mathbb{R}$. Then w extends to a non-vanishing solution of (4.2) on all of $\mathbb{R}$. Furthermore, any solution of $u'' + pu = 0$ vanishes on a given interval $[\tau_1, \tau_2]$ at most $N + 1$ times, with

$$N \;=\; \left[\, \frac{1}{2\pi}\int_{\tau_1}^{\tau_2} w(s)^{-2} ds \right] \;. \tag{4.3}$$

Proof. Suppose, say, $w > 0$ on J. Fix $s_0 \in J$ and let u_0 be a solution of $u''+pu = 0$ for which $u_0(s_0) > 0$. For s near s_0 we can write

$$w = \rho\, u_0\,,$$

where ρ is positive. Equation (4.2) then becomes

$$u_0\rho'' + 2u_0'\rho' = \frac{1}{4}u_0^{-3}\rho^{-3}\,,$$

which corresponds to

$$u_0^2\left(u_0^2\rho'\right)' = \frac{1}{4}\rho^{-3}\,.$$

On the interval around s_0 where $u_0 > 0$ we consider the change of variable $x = x(s)$ for which $d/dx = u_0^2\, d/ds$, that is,

$$x(s) = \int_{s_0}^{s} u_0^{-2}(\tau)d\tau\,. \tag{4.4}$$

We see that the function

$$\lambda(x) = \rho(s(x))$$

is a solution of (4.1), which by Lemma 1, is defined and nonzero for all x. This proves that $w(s)$ extends as a positive solution of (4.2) as long as u_0 remains positive.

In order to show that w can be continued as a positive solution of (4.2) for all s, we must analyze the behavior of ρu_0 as s approaches a first zero of u_0 to the left or right of s_0. We give the argument for $s > s_0$, the case $s < s_0$ being handled similarly. Suppose that $u_0(s_1) = 0$ for the first time at some $s_1 > s_0$. Then

$$x(s) \sim \frac{c_1}{s - s_1} \quad , \quad s \to s_1^-\,,$$

and so

$$\lim_{s\to s_1^-} x(s) \to \infty\,.$$

Since $\lim_{x\to\infty}\lambda(x) \to \infty$ at a linear rate, we see that $\rho(s) \sim c_2/(s_1 - s)$ as $s \to s_1^-$, from which it follows that

$$\lim_{s\to s_1^-}(\rho u_0)(s) \;=\; \alpha$$

exists and is positive. Direct integration of (4.2) shows that $w'(s)$ has a limit as $s \to s_1^-$, proving that w can be continued as a positive solution across $s = s_1$. This argument can be repeated infinitely often if necessary, and proves that w extends to a non-vanishing solution of (4.2) defined for all s.

Let $J = [\tau_1, \tau_2]$ be any finite interval and let w be a solution of (4.2). Let u_0 be a non-trivial solution of $u'' + pu = 0$ with $u_0(\tau_1) = 0$. By the preceding argument, we can write $w = \rho u_0$, allowing for cancelations at the zeros of u_0. It s_1, s_2 are consecutive zeros of u_0 we see that

$$\int_{s_1}^{s_2} w(s)^{-2}ds \;=\; \int_{s_1}^{s_2}[(\rho u_0)(s)]^{-2}ds \;=\; \int_{-\infty}^{\infty}\lambda(x)^{-2}dx \;=\; 2\pi\,.$$

This shows that the number of zeros of u_0 in the interval $[\tau_1, \tau_2]$ does not exceed $N+1$, with N given by (4.3). By Sturm's interlacing theorem, the same conclusion holds for any solution of $u'' + pu = 0$. This finishes the proof.

□

In our final result we will consider equation (4.2) with $p = \frac{1}{4}k^2$. Let Z be the set of zeros of a solution of

$$u'' + \frac{1}{4}k^2 u = 0\,. \tag{4.5}$$

We call (4.5) *oscillatory* if Z is unbounded from above and below, *oscillatory at* ∞ if Z is bounded only from below, *oscillatory at* $-\infty$ if Z is bounded only from above, and *non-oscillatory* if Z is bounded.

Theorem 4.3. *There exists a parametrization $\psi : I \to \Gamma$ which is onto and for which $S_1\psi = \frac{1}{2}$ everywhere. The interval I is finite, of the form (a, ∞), $(-\infty, b)$ or equal to $\mathbb{R}$ according to whether the equation $u'' + \frac{1}{4}k^2 = 0$ is not oscillatory, oscillatory at ∞, oscillatory at $-\infty$, or oscillatory.*

Proof. Let w be any solution of (4.2) with $p(s) = \frac{1}{4}k^2(s)$, and let

$$t = h(s) = \int_{s_0}^{s} w(\tau)^{-2} d\tau\,.$$

Because $w'' + \frac{1}{4}(k^2 - w^{-4})w = 0$ we have that

$$Sh = \frac{1}{2}\left(k^2 - w^{-4}\right)\,.$$

Let ψ be defined by

$$\psi(t) = \phi(s(t))\,.$$

Then $v = |\psi'| = w^2 = (h')^{-1}$ and

$$S_1\psi = Ss + \frac{1}{2}w^4k^2 = -(Sh)(h')^{-2} + \frac{1}{2}w^4k^2 = \frac{1}{2}\,. \tag{4.6}$$

On the other hand, it follows from Theorem 2.5 that $\lim_{s\to\infty} x(s) = \infty$ exactly when (4.5) is oscillatory at ∞, and similarly with the limit as $s \to \infty$. This finishes the proof. □

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